

The packing chromatic number of the infinite square lattice is ≤ 16

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Abstract

Using a SAT-solver on top of a partial previously-known solution we improve the upper bound of the packing chromatic number of the infinite square lattice from 17 to 16. And then to 15.

1 Introduction

The notion of *packing colouring* comes from the area of frequency planning in wireless networks, and was introduced by Goddard et al. in [4] under the name *broadcast colouring*. The *packing chromatic number* of a graph G , denoted by $\chi_p(G)$, is the smallest integer k such that $V(G)$ can be partitioned into k disjoint sets X_1, \dots, X_k , where for each pair of vertices $x, y \in X_i$ the minimum distance between them in G , $\text{dist}_G(x, y)$, is greater than i , for each $i \in \{1, \dots, k\}$. In other words, vertices with the same colour i are pairwise at distance greater than i .

The infinite square lattice $P_{\mathbb{Z}} \square P_{\mathbb{Z}}$ is the graph with vertex set $\mathbb{Z} \times \mathbb{Z}$ and edge set

$$\{((x_1, y_1), (x_2, y_2)) : (x_1 = x_2 \wedge |y_1 - y_2| = 1) \vee (y_1 = y_2 \wedge |x_1 - x_2| = 1)\}.$$

If $P_{\mathbb{Z}}$ is the graph with vertices \mathbb{Z} and edges (x, y) given by $|x - y| = 1$, then the infinite square lattice is the Cartesian product $P_{\mathbb{Z}} \square P_{\mathbb{Z}}$, which explains our notation.

The packing chromatic number of the infinite square lattice has been the topic of a number of papers. Goddard et al. showed in [4] that $\chi_p(P_{\mathbb{Z}} \square P_{\mathbb{Z}})$ is finite, more precisely between 9 and 23 (inclusive). In contrast, the packing chromatic number of the infinite triangular lattice is infinite [3], though the packing chromatic number of the infinite hexagonal lattice is 7 [5]. The upper bound of [4] is witnessed by a finite grid which can be eternally translated in order to periodically cover the plane.

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Fiala and Lidický [2] then improved the lower bound to 10, and Schwenk [6] improved the upper bound to 22. Later, Ekstein, Fiala, Holub and Lidický used a computer to improve the lower bound to 12 [1] and Soukal and Holub used a clever Simulated Annealing algorithm to improve the upper bound to 17 [7]. Thus these last bounds, in contrast to those that went before, both made fundamental use of mechanical computation.

The first author heard of this problem at a talk by Bernard Lidický at the 8th Slovenian Conference on Graph Theory (Bled 2011). While this problem may not be important, few who worked on it can doubt that it is very addictive, and further provides a vehicle through which to ponder different algorithmic techniques. One of the curiosities of the problem is that we have little theoretical insight into it. For example, suppose there is a packing colouring with k colours:

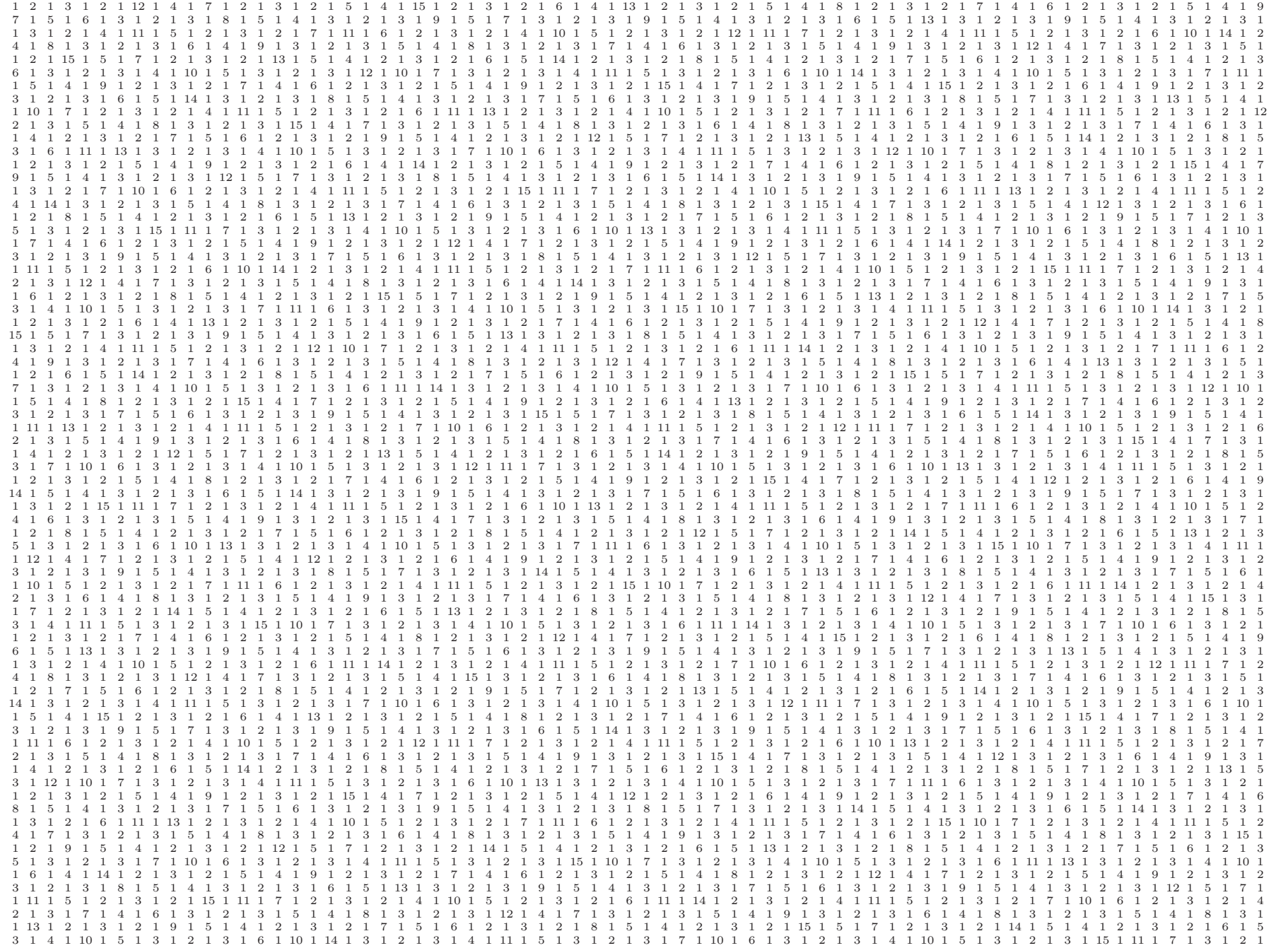
- does there exist an m together with an $m \times m$ grid that witnesses a periodic packing colouring with k ?
- does there exist a packing colouring with k that has colour 1 at maximal density $(1/2)$ asymptotically?
- does there exist a packing colouring with k so that, for $i < j \leq k$ the asymptotic frequency of colour i is no more than the asymptotic frequency of j ?

In the present note we improve the upper bound from 17 to 16 and then to 15. As with all these upper bounds we give a periodic packing colouring based on a finite grid. We make essential use of the periodic 24×24 17-colouring given in [7]. For our 16-colouring, we take their colouring and remove colours 8 to 17, then blow this up from 24×24 to 48×48 . We then give the resulting partially coloured grid to a SAT solver to see if a 16-colouring is possible, which it turns out it is. Our 16-colouring is specified as the obvious periodic translation of the colouring in Figure 1. Our method with a SAT solver can not run efficiently unless many colours are planted (for example, with just colour 1 planted on a 24×24 grid we made no progress). On the other hand, we believe Simulated Annealing might well not find our solution (without a high temperature parameter) and one argument is furnished for this by the tables in Figure 2. For the Soukal-Holub 17-colouring, the frequency of each colour monotonely decreases as the colour number rises (something quite unsurprising). Whereas for our 16-colouring, this monotonicity is broken in two places. Our 15-colouring is, however, monotone.

For our 15-colouring, we take their colouring and remove colours 5 to 17, then blow this up from 24×24 to 72×72 . We then give the resulting partially coloured grid to a SAT solver to see if a 15-colouring is possible, which it is. Here we make use of the SAT-solving technique of commander variables.

For the 16-colouring, we used the SAT-solver Sat4j and our instance ran for approximately two days on a computer comprising CPU Intel(R) Xeon(R), CPU E5-1620 v2 @ 3.70GHz (4 cores, 8 Hyperthreaded) with 24 GB RAM. Note that Sat4j operates on a single core.

1	7	1	4	1	6	1	3	1	2	1	3	1	16	1	4	1	5	1	3	1	2	1	3	1	7	1	4	1	6	1	3	1	2	1	3	1	13	1	4	1	5	1	3	1	2	1	3							
2	1	3	1	2	1	5	1	7	1	4	1	2	1	3	1	2	1	9	1	6	1	14	1	2	1	3	1	2	1	5	1	7	1	4	1	2	1	3	1	2	1	14	1	6	1	8	1							
1	5	1	13	1	3	1	2	1	3	1	6	1	5	1	7	1	3	1	2	1	3	1	4	1	5	1	10	1	3	1	2	1	3	1	2	1	3	1	5	1	7	1	3	1	2	1	3	1	4					
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1	6	1	3	1	2	1	3	1	5	1	8	1	4	1	3	1	2	1	3	1	7	1	8	1	6	1	3	1	2	1	3	1	5	1	16	1	4	1	3	1	2	1	3	1	7	1	9							
2	1	4	1	5	1	11	1	2	1	3	1	2	1	13	1	5	1	11	1	2	1	3	1	2	1	4	1	5	1	11	1	2	1	3	1	2	1	9	1	5	1	11	1	2	1	3	1							
1	3	1	2	1	3	1	6	1	4	1	7	1	3	1	2	1	3	1	15	1	4	1	5	1	3	1	2	1	3	1	6	1	4	1	7	1	3	1	2	1	3	1	12	1	4	1	5							
15	1	7	1	8	1	2	1	3	1	2	1	5	1	6	1	4	1	2	1	3	1	2	1	9	1	7	1	13	1	2	1	3	1	2	1	5	1	6	1	4	1	2	1	3	1	2	1							
1	2	1	3	1	4	1	5	1	12	1	3	1	2	1	3	1	7	1	5	1	6	1	3	1	2	1	3	1	4	1	5	1	10	1	3	1	2	1	3	1	2	1	7	1	5	1	6	1	3					
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1	3	1	6	1	9	1	7	1	3	1	2	1	3	1	5	1	8	1	4	1	3	1	2	1	3	1	6	1	14	1	7	1	3	1	2	1	3	1	5	1	13	1	4	1	3	1	2							
11	1	2	1	3	1	2	1	4	1	5	1	14	1	2	1	3	1	2	1	3	1	2	1	12	1	5	1	11	1	2	1	3	1	2	1	3	1	5	1	11	1	2	1	3	1	2	1	9	1	5	1			
1	16	1	4	1	5	1	3	1	2	1	3	1	7	1	4	1	6	1	3	1	2	1	3	1	8	1	4	1	5	1	3	1	2	1	3	1	7	1	4	1	6	1	3	1	2	1	3	1	2	1	3			
2	1	3	1	2	1	13	1	6	1	8	1	2	1	3	1	2	1	5	1	7	1	4	1	2	1	3	1	2	1	3	1	9	1	1	2	1	3	1	2	1	12	1	2	1	3	1	2	1	5	1	7	1	4	1
1	5	1	7	1	3	1	2	1	3	1	4	1	5	1	11	1	3	1	2	1	3	1	6	1	5	1	7	1	3	1	2	1	3	1	1	2	1	3	1	4	1	5	1	10	1	3	1	2	1	3	1	6		
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1	6	1	3	1	2	1	3	1	5	1	14	1	4	1	3	1	2	1	3	1	1	2	1	7	1	9	1	6	1	3	1	2	1	3	1	1	2	1	3	1	5	1	12	1	4	1	3	1	2	1	15			
2	1	4	1	5	1	11	1	2	1	3	1	2	1	9	1	5	1	11	1	2	1	3	1	2	1	10	1	4	1	5	1	15	1	2	1	3	1	1	2	1	3	1	2	1	13	1	4	1	3	1	2	1		
1	3	1	2	1	3	1	6	1	4	1	7	1	3	1	2	1	3	1	13	1	4	1	5	1	3	1	2	1	3	1	6	1	4	1	7	1	3	1	2	1	3	1	1	2	1	3	1	14	1	4	1	5		
9	1	7	1	8	1	2	1	3	1	2	1	3	1	2	1	5	1	6	1	4	1	2	1	3	1	2	1	3	1	2	1	3	1	1	2	1	3	1	2	1	5	1	6	1	4	1	2	1	3	1	2	1		
1	2	1	3	1	4	1	5	1	10	1	3	1	2	1	3	1	7	1	5	1	6	1	3	1	2	1	3	1	4	1	5	1	11	1	3	1	2	1	3	1	7	1	5	1	6	1	3	1	2	1	3	1		
4	1	5	1	2	1	3	1	2	1	15	1	4	1	12	1	2	1	3	1	1	2	1	10	1	4	1	5	1	2	1	3	1	1	2	1	3	1	2	1	10	1	4	1	9	1	2	1	3	1	2	1	10	1	
1	3	1	6	1	16	1	7	1	3	1	2	1	3	1	5	1	8	1	4	1	3	1	2	1	3	1	6	1	9	1	7	1	3	1	2	1	3	1	1	2	1	3	1	5	1	8	1	4	1	3	1	2		
11	1	2	1	3	1	2	1	4	1	5	1	11	1	2	1	3	1	2	1	3	1	1	2	1	9	1	5	1	11	1	2	1	3	1	1	2	1	3	1	2	1	16	1	7	1	3	1	2	1	12	1	5	1	
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1	5	1	7	1	3	1	2	1	3	1	4	1	5	1	10	1	3	1	2	1	3	1	1	2	1	6	1	5	1	7	1	3	1	2	1	3	1	1	2	1	3	1	5	1										

Figure 3: Our 72×72 periodic solution in 15 colours

2 Final Remarks

We improved the bound for the packing chromatic number of the infinite square lattice by using a SAT-solver on top of partial result garnered by Simulated Annealing. This mixed method approach succeeded where we are unconvinced either approach by itself would have. Several years ago, Bernard Lidický indicated to the first author that most people who worked on this problem believed the true answer to be closer to 12 than to 17. Our view does not contradict this but we see no reason why the answer should be 16 (indeed, 15). We will endeavour to improve this upper bound but we will not be surprised if we fail.

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1	7	1	4	1	6	1	3	1	2	1	3	1	—	1	4	1	5	1	3	1	2	1	3
2	1	3	1	2	1	5	1	7	1	4	1	2	1	3	1	2	1	—	1	6	1	—	1
1	5	1	—	1	3	1	2	1	3	1	6	1	5	1	7	1	3	1	2	1	3	1	4
3	1	2	1	—	1	4	1	—	1	2	1	3	1	2	1	—	1	4	1	5	1	2	1
1	6	1	3	1	2	1	3	1	5	1	—	1	4	1	3	1	2	1	3	1	7	1	—
2	1	4	1	5	1	—	1	2	1	3	1	2	1	—	1	5	1	—	1	2	1	3	1
1	3	1	2	1	3	1	6	1	4	1	7	1	3	1	2	1	3	1	—	1	4	1	5
—	1	7	1	—	1	2	1	3	1	2	1	5	1	6	1	4	1	2	1	3	1	2	1
1	2	1	3	1	4	1	5	1	—	1	3	1	2	1	3	1	7	1	5	1	6	1	3
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1	3	1	6	1	—	1	7	1	3	1	2	1	3	1	5	1	—	1	4	1	3	1	2
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1	—	1	4	1	5	1	3	1	2	1	3	1	7	1	4	1	6	1	3	1	2	1	3
2	1	3	1	2	1	—	1	6	1	—	1	2	1	3	1	2	1	5	1	7	1	4	1
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1	4	1	3	1	2	1	3	1	7	1	—	1	6	1	3	1	2	1	3	1	5	1	—
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5	1	6	1	4	1	2	1	3	1	2	1	—	1	7	1	—	1	2	1	3	1	2	1
1	2	1	3	1	7	1	5	1	6	1	3	1	2	1	3	1	4	1	5	1	—	1	3
4	1	—	1	2	1	3	1	2	1	—	1	4	1	5	1	2	1	3	1	2	1	—	1
1	3	1	5	1	—	1	4	1	3	1	2	1	3	1	6	1	—	1	7	1	3	1	2
—	1	2	1	3	1	2	1	—	1	5	1	—	1	2	1	3	1	2	1	4	1	5	1

Figure 4: The Soukal-Holub 17-colouring on 24×24 with colours 8 to 17 removed.